

2-KNOTS WITH SOLVABLE GROUP

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ABSTRACT. We complete the TOP classification of 2-knots with torsion-free, solvable knot group by showing that fibred 2-knots with closed fibre the Hantzsche-Wendt flat 3-manifold HW are not reflexive, while every fibred 2-knot with closed fibre a Nil -manifold with base orbifold $S(3,3,3)$ is reflexive, and by giving explicit normal forms for the strict weight orbits of normal generators for the groups of all knots in either class. We also determine when the knots are amphicheiral or invertible, and show that the only doubly null-concordant knots with such groups are those with group $\pi\tau_2 9_{46}$.

The largest class of groups π over which TOP surgery techniques in dimension 4 are known to hold is the class SA obtained from groups of subexponential growth by extensions and increasing unions. No such group has a noncyclic free subgroup. The known 2-knot groups in this class are either torsion-free and solvable or have finite commutator subgroup. (It seems plausible that there may be no others. See Theorem 15.13 of [5] and §4 below for evidence in this direction.)

If the group of a nontrivial 2-knot K is torsion-free and elementary amenable then K is either the Fox knot (Example 10 of [2]) or is fibred, with closed fibre $\mathbb{R}^3/\mathbb{Z}^3$, the Hantzsche-Wendt flat 3-manifold $HW = \mathbb{R}^3/G_6$ or a Nil^3 -manifold. (See Theorem 2.) Each such knot is determined up to Gluck reconstruction, TOP isotopy and change of orientations by its group π and weight orbit (the orbit of a weight element under the action of $Aut(\pi)$). This orbit is unique for the Fox knot and for the fibred knots with closed fibre $\mathbb{R}^3/\mathbb{Z}^3$ (the Cappell-Shaneson knots) or a coset space of the Lie group Nil . In each of these cases the questions of amphicheirality, invertibility and reflexivity have been decided. (See [5, 6, 7].)

There are just two possible knot groups $G(\pm)$ realized by knots with closed fibre HW . No such knot is reflexive. The 3-twist spin of the figure eight knot $\tau_3 4_1$ and its Gluck reconstruction $\tau_3 4_1^*$ (with group $G(+)$) are strongly \pm amphicheiral but not strongly invertible. The remaining

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knots have closed fibre the 2-fold branched cover of S^3 , branched over a Montesinos knot $k(e, \eta) = K(0|e; (3, \eta), (3, 1), (3, 1))$, with e even and $\eta = \pm 1$. These are all reflexive. This class includes the 2-twist spins $\tau_2 k(e, \eta)$ of such Montesinos knots, which are strongly +amphicheiral but not invertible. With the exception of four other knots with group $G(-)$ none of the other knots is amphicheiral or invertible. In all cases we give explicit normal forms for the strict weight orbits. This completes the TOP classification of such knots. (However, it is not known whether they are all smooth knots in the standard S^4 .)

When the commutator subgroup of a 2-knot group is finite the list of possible groups and weight orbits is known, but the surgery obstruction groups are large, and there are in general infinitely many TOP locally flat knots with a given such group. Thus it is reasonable to restrict attention to those which are fibred. The closed fibre is then a spherical manifold S^3/π' . In this case the question of reflexivity has been settled for 10 of the 17 possible families of such knots [15]. It is likely that none of the remaining knots are reflexive, but this has not yet been confirmed.

In the final section we show that a nontrivial 2-knot K such that πK is torsion-free and solvable is TOP doubly slice if and only if $\pi K \cong \pi \tau_2 9_{46}$. We observe also that if K is 2-knot such that $(\pi K)'$ is finite then it is TOP doubly slice if and only if $\pi K \cong \pi \tau_5 3_1 \cong Z \times I^*$.

1. KNOT GROUPS

If G is a group then G' and ζG shall denote the commutator subgroup and the centre of G , while $C_G(H)$ and $N_G(H)$ shall denote the centralizer and normalizer of the subgroup $H \leq G$, respectively.

An automorphism ϕ of a group G is *meridional* if $\langle \langle g^{-1}\phi(g) \mid g \in G \rangle \rangle_G = G$. When G is finitely generated and solvable this holds if and only if $H_1(\phi) - 1$ is an automorphism of the abelianization $H_1(G)$. If ϕ and ψ are meridional automorphisms of G then the semidirect products $G \rtimes_{\phi} Z$ and $G \rtimes_{\psi} Z$ are isomorphic if and only if the outer automorphism class $[\phi]$ is conjugate to $[\psi]$ or $[\psi]^{-1}$ in $Out(G)$. There is an isomorphism preserving the stable letters of the HNN extensions if and only if ϕ and ψ are conjugate in $Aut(G)$. (See Lemma 1.1 of [5].)

Let $t \in \pi = G \rtimes_{\phi} Z$ be an element whose normal closure $\langle \langle t \rangle \rangle_{\pi}$ is the whole group. Every such “weight element” w is of the form $w = gt$ or $w = (gt)^{-1}$, for some $g \in G$. The strict weight orbit of w is the set $\{\alpha(w) \mid \alpha \in Aut(\pi), \alpha(w) \equiv w \text{ mod } G\}$.

If π has a weight element then $G = \pi'$. Let $c_x \in Aut(G)$ be the automorphism induced by conjugation by x in π . Two weight elements

t and gt with $g \in G$ are in the same strict weight orbit if and only if there is an automorphism ψ of G such that $c_g = \psi \cdot c_t \cdot \psi^{-1} c_t^{-1}$, by Theorem 14.1(3) of [5]. In particular, c_t and c_{gt} have the same order, and $[\psi][\phi] = [\phi][\psi]$ in $\text{Out}(G)$.

If π is a solvable n -knot group then π'' acts transitively on the set of weight elements representing a given generator t of π/π' , by Theorem 14.1(4) of [5]. Unfortunately, this action does not usually induce an action on the set of weight orbits. This result does not extend to all virtually solvable groups. In particular, it does not hold for $Z \times I^*$.

Although it is possible to study the automorphism groups considered below by purely algebraic means, we shall use embeddings in the appropriate affine groups to guide the construction of homeomorphisms and isotopies.

2. SELF-HOMEOMORPHISMS OF KNOT EXTERIORS

We assume that the spheres S^n are oriented. Let $K : S^2 \rightarrow S^4$ be a 2-knot with exterior X , and fix a homeomorphism $\partial X \cong S^2 \times S^1$ which is compatible with the orientations of the spheres S^1 , S^2 and $X \subset S^4$. Let $\tau(x, y) = (\rho(y)(x), y)$ for all (x, y) in $S^2 \times S^1$, where $\rho : S^1 \rightarrow SO(3)$ is an essential map. The *Gluck reconstruction* of K is the knot K^* given by the composite inclusion

$$S^2 \subset S^2 \times D^2 \subset X \cup_\tau S^2 \times D^2 \cong S^4.$$

The knot K is *reflexive* if K^* is isotopic to one of the four knots $K, r_4 K, K r_2$ or $-K = r_4 K r_2$ obtained by composition with reflections r_n of S^n .

We may extend any self-homeomorphism h of X “radially” to a self-homeomorphism of the knot manifold $M(K) = X \cup D^3 \times S^1$ which maps the cocore $C = \{0\} \times S^1$ to itself. If h preserves both orientations or reverses both orientations then it fixes the meridian, and we may assume that $h|_C = id_C$. If h reverses the meridian t , we may still assume that it fixes a point on C . We take such a fixed point as the basepoint for $M(K)$. Let h'_* be the induced automorphism of π' .

If K is invertible or \pm amphicheiral there is a self-homeomorphism h of (S^4, K) which changes the orientations appropriately, but does not twist the normal bundle of $K(S^2) \subset S^4$. If it is reflexive there is such a self-homeomorphism which changes the framing of the normal bundle. Thus if K is $-$ amphicheiral there is such an h which reverses the orientation of $M(K)$ and h'_* commutes with the meridional automorphism c_t . If K is invertible or $+$ amphicheiral there is a homeomorphism h such that $h'_* c_t h'_* = c_t^{-1}$ and which preserves or reverses the orientation.

3. SECTIONS OF THE MAPPING TORUS

Let θ be a self-homeomorphism of a 3-manifold F , with mapping torus $M(\theta) = F \times [0, 1] / \sim$, where $(f, 0) \sim (\theta(f), 1)$ for all $f \in F$, and canonical projection $p_\theta : M(\theta) \rightarrow S^1$, given by $p_\theta([f, s]) = e^{2\pi i s}$ for all $[f, s] \in M(\theta)$. The mapping torus $M(\theta)$ is orientable if and only if θ is orientation-preserving. If $\theta' = h\theta h^{-1}$ for some self-homeomorphism h of F then $[f, s] \mapsto [h(f), s]$ defines a homeomorphism $m(h) : M(\theta) \rightarrow M(\theta')$ such that $p_{\theta'} m(h) = p_\theta$. Similarly, if θ' is isotopic to θ then $M(\theta') \cong M(\theta)$.

If $P \in F$ is fixed by θ then the image of $P \times [0, 1]$ in $M(\theta)$ is a section of p_θ . In particular, if the fixed point set of θ is connected there is a canonical isotopy class of sections. If moreover $\theta_* = \pi_1(\theta)$ is meridional these determine a preferred conjugacy class of weight elements in the group $\pi_1(M(\theta))$. (Two sections are isotopic if and only if they represent conjugate elements of π .)

In general, we may isotope θ to have a fixed point P . Let $t \in \pi_1(M(\theta))$ correspond to the constant section of $M(\theta)$, and let $u = gt$ with $g \in \pi_1(F)$. Let $\gamma : [0, 1] \rightarrow F$ be a loop representing g . There is an isotopy h_s from $h_0 = id_F$ to $h = h_1$ which drags P around γ , so that $h_s(P) = \gamma(s)$ for all $0 \leq s \leq 1$. Then $H([f, s]) = [(h_s)^{-1}(f), s]$ defines a homeomorphism $M(\theta) \cong M(h^{-1}\theta)$. Under this homeomorphism the constant section of $p_{h^{-1}\theta}$ corresponds to the section of p_θ given by $m_u(t) = [\gamma(t), t]$, which represents u . If F is a geometric 3-manifold we may assume that γ is a geodesic path.

Suppose henceforth that θ is orientation-preserving and θ_* is meridional. Then surgery on a section gives a 2-knot. There are two possible framings for the surgery, but the exteriors of the two knots are homeomorphic.

This is the situation for twist-spins, where F is a cyclic branched cover of S^3 , branched over a classical knot, and θ generates the covering group. The subset fixed by θ is connected and nonempty, since it is the branch locus. The knot exterior is the complement of an open regular neighbourhood of the canonical section of the mapping torus of θ .

If F has universal cover $\tilde{F} \cong \mathbb{R}^3$ and h is a self-homeomorphism of $M(\theta)$ which fixes a section setwise the behaviour of h with respect to the orientations is detected by the effect of h'_* on $H_3(F; \mathbb{Z})$ and whether $h'_* c_t h'_* = c_t$ or c_t^{-1} . As in [1, 7] (and Chapter 18 of [5]), in order to determine whether h changes the framing it shall suffice to pass to the irregular covering space $M(\tilde{\theta}) = \tilde{F} \times_{\tilde{\theta}} S^1$. We seek a coordinate homeomorphism $\tilde{F} \cong \mathbb{R}^3$ which gives convenient representations of the

maps in question, and then use an isotopy from the identity to $\tilde{\theta}$ to identify $M(\tilde{\theta})$ with $\mathbb{R}^3 \times S^1$.

Lemma 1. *Let K be a fibred 2-knot. If there is a self homeomorphism h of $X(K)$ which is the identity on one fibre and such that $h|_{\partial X} = \tau$ then all knots \tilde{K} with $M(\tilde{K}) \cong M(K)$ are reflexive. In particular, this is so if the monodromy of K has order 2.*

Proof. We may extend h to a self-homeomorphism \hat{h} of $M(K)$ which fixes the surgery cocore $C \cong S^1$. After an isotopy of h , we may assume that it is the identity on a product neighbourhood $N = \hat{F} \times [-\epsilon, \epsilon]$ of the closed fibre. Since any weight element for π may be represented by a section γ of the bundle which coincides with C outside N , we may use h to change the framing of the normal bundle of γ for any such knot. Hence every such knot is reflexive.

If the monodromy of K has order 2 then “turning the mapping torus upside-down” changes the framing of the normal bundle and fixes one fibre. \square

The reflexivity of 2-twist spins is due to Litherland. See [12, 13, 14].

4. TORSION-FREE ELEMENTARY AMENABLE IMPLIES SOLVABLE

We shall let Φ denote the group of the Fox knot. This is an ascending HNN extension $\Phi \cong Z *_2$, with presentation $\langle a, t \mid tat^{-1} = a^2 \rangle$.

Theorem 2. *Let K be a 2-knot whose group $\pi = \pi K$ is torsion-free and elementary amenable. Then K is trivial, the Fox knot, or is fibred with closed fibre a flat 3-manifold or a Nil-manifold.*

Proof. If π is torsion-free and has more than one end then $\pi \cong Z$, and so K is trivial [3]. If π has one end and $H^2(\pi; \mathbb{Z}[\pi]) = 0$ then $M(K)$ is aspherical, by Theorem 3.5 of [5], and so $H^4(\pi; \mathbb{Z}[\pi]) \neq 0$. Otherwise $H^2(\pi; \mathbb{Z}[\pi]) \neq 0$. In all cases $H^s(\pi; \mathbb{Z}[\pi]) \neq 0$ for some $s \leq 4$, and so π is virtually solvable, by Proposition 3 of [9]. It then follows that either $\pi \cong Z$ or $\pi \cong \Phi = Z *_2$, or that π is virtually poly- Z of Hirsch length 4. (See Theorem 15.13 of [5].) If $\pi \cong \Phi$ then K is the Fox knot or its reflection [6], while the remaining cases are covered in Chapter 16 of [5]. \square

Can we relax the condition on torsion? Let π be an elementary amenable knot group. Since π is finitely presentable and has an infinite cyclic quotient it is an HNN extension with finitely generated base and associated subgroups. Since it has no noncyclic free subgroups the HNN extension is ascending: $\pi \cong H *_\phi$, where H is finitely generated and

$\phi : H \rightarrow H$ is injective. If moreover H is FP_3 and virtually indicable then either π' is finite or π is torsion-free, by Theorem 15.13 of [5].

The additional hypotheses on H could be removed if we had a better understanding of when $H^2(\pi; \mathbb{Z}[\pi]) = 0$. Suppose that whenever G is a finitely presentable group such that either

- (1) G has an elementary amenable normal subgroup E such that
 - (a) $h(E) > 2$; or
 - (b) $h(E) = 2$ and G/E is infinite; or
 - (c) $h(E) = 1$ and G/E has one end; or
- (2) $G \cong B *_\phi$ is an ascending HNN extension with finitely generated, 1-ended base B ;

then $H^2(G; \mathbb{Z}[G]) = 0$.

We may then argue as follows. Since π is finitely presentable and has an infinite cyclic quotient it is an HNN extension with finitely generated base and associated subgroups. Since it has no noncyclic free subgroups the HNN extension is ascending: $\pi \cong H *_\phi$, where H is finitely generated and $\phi : H \rightarrow H$ is injective. Since π is elementary amenable and infinite $\beta_1^{(2)}(\pi) = 0$. If $h(\pi) = 1$ then π' is finite. Suppose that π' is infinite. Then π has one end. If $h(\pi) > 2$ or $h(\pi) = 2$ and the HNN base H has one end then $H^2(\pi; \mathbb{Z}[\pi]) = 0$ and so the knot manifold $M(K)$ is aspherical, by Theorem 3.5 of [5]. Hence π is torsion-free and virtually solvable, by Theorem 1.11 of [5]. (Closer examination shows that it must be polycyclic. See Chapter 16 of [5].) Otherwise H must have two ends. Let T be the maximal finite normal subgroup of H . Then $\phi(T) = T$, since ϕ is injective, and so T is normal in π . Hence $T = 1$ and $\pi \cong \Phi$, by Theorem 15.2 of [5].

5. THE HANTZSCHE-WENDT FLAT 3-MANIFOLD

The group of affine motions of 3-space is $Aff(3) = \mathbb{R}^3 \rtimes GL(3, \mathbb{R})$. The action is given by $(v, A)(x) = Ax + v$, for all $x \in \mathbb{R}^3$. Therefore $(v, A)(w, B) = (v + Aw, AB)$.

Let $\{e_1, e_2, e_3\}$ be the standard basis of \mathbb{R}^3 , and let $X, Y, Z \in GL(3, \mathbb{Z})$ be the diagonal matrices $X = \text{diag}[1, -1, -1]$, $Y = \text{diag}[-1, 1, -1]$ and $Z = \text{diag}[-1, -1, 1]$. Let $x = (\frac{1}{2}e_1, X)$, $y = (\frac{1}{2}(e_2 - e_3), Y)$ and $z = (\frac{1}{2}(e_1 - e_2 + e_3), Z)$. The subgroup of $Aff(3)$ generated by x and y is the Hantzsche-Wendt flat 3-manifold group G_6 , with presentation

$$\langle x, y, z \mid xy^2x^{-1}y^2 = yx^2y^{-1}x^2 = 1, z = xy \rangle.$$

The translation subgroup $T = G_6 \cap \mathbb{R}^3$ is free abelian, with basis $\{x^2, y^2, z^2\}$. (This is the maximal abelian normal subgroup of G_6 .) The holonomy group $H = \{I, X, Y, Z\} \cong (Z/2Z)^2$ is the image of

G_6 in $GL(3, \mathbb{R})$. (Thus $H \cong G_6/T$.) We may clearly take $\{1, x, y, z\}$ as coset representatives for H in G_6 . The commutator subgroup G'_6 is free abelian, with basis $\{x^4, y^4, x^2y^2z^{-2}\}$. Thus $2T < G'_6 < T$, $T/G'_6 \cong (Z/2Z)^2$ and $G'_6/2T \cong Z/2Z$.

The orbit space $HW = G_6 \backslash \mathbb{R}^3$ is the *Hantzsche-Wendt* flat 3-manifold.

6. THE AUTOMORPHISM GROUP OF G_6

Every automorphism of the fundamental group of a flat n -manifold is induced by conjugation in $Aff(n)$, by a theorem of Bieberbach. Hence $Aut(G_6) \cong N/C$ and $Out(G_6) \cong N/CG_6$, where $C = C_{Aff(3)}(G_6)$ and $N = N_{Aff(3)}(G_6)$.

If $(v, A) \in Aff(3)$ commutes with all elements of G_6 then $AB = BA$ for all $B \in H$, so A is diagonal, and $v + Aw = w + Bv$ for all $(w, B) \in G_6$. Taking $B = I$, we see that $Aw = w$ for all $w \in \mathbb{Z}^3$, so $A = I$, and then $v = Bv$ for all $B \in H$, so $v = 0$. Thus $C = 1$, and so $Aut(G_6) \cong N$.

If $(v, A) \in N$ then $A \in N_{GL(3, \mathbb{R})}(H)$ and A preserves $T = \mathbb{Z}^3$, so $A \in N_{GL(3, \mathbb{Z})}(H)$. Therefore $W = AXA^{-1}$ is in H . Hence $WA = AX$ and so $Wae_1 = Ae_1$ is up to sign the unique basis vector fixed by W . Applying the same argument to AYA^{-1} and AZA^{-1} , we see that $N_{GL(3, \mathbb{R})}(H)$ is the group of “signed permutation matrices”, generated by the diagonal matrices and permutation matrices. Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If A is a diagonal matrix in $GL(3, \mathbb{Z})$ then $(0, A) \in N$. Thus $\tilde{a} = (0, -X)$, $\tilde{b} = (0, -Y)$ and $\tilde{c} = (0, -Z)$ are in N . It is easily seen that $N \cap \mathbb{R}^3 = \frac{1}{2}\mathbb{Z}^3$, with basis $\tilde{d} = (\frac{1}{2}e_1, I)$, $\tilde{e} = (\frac{1}{2}e_2, I)$ and $\tilde{f} = (\frac{1}{2}e_3, I)$. It is also easily verified that $\tilde{i} = (-\frac{1}{4}e_3, P)$ and $\tilde{j} = (\frac{1}{4}(e_1 - e_2), J)$ are in N , and that N is generated by $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{i}, \tilde{j}\}$.

The natural action of N on \mathbb{R}^3 is isometric, since $N_{GL(3, \mathbb{R})}(H) < O(3)$, and so N/G_6 acts isometrically on the orbit space HW . In fact every isometry of HW lifts to an affine transformation of \mathbb{R}^3 which normalizes G_6 , and so $Isom(HW) \cong Out(G_6)$. The isometries which preserve the orientation are represented by pairs (v, A) with $\det(A) = 1$. (Thus $\tilde{a}, \tilde{b}, \tilde{c}$ and \tilde{j} represent orientation reversing isometries.)

In Chapter 8.§2 of [5] we showed that $Aut(G_6)$ is generated by the automorphisms a, b, c, d, e, f, i and j which send x to $x^{-1}, x, x, x, y^2x, z^2x, y, z$ and y to $y, y^{-1}, z^2y, x^2y, y, z^2y, x, x$, respectively. These automorphisms are induced by conjugation by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{i}$ and \tilde{j} , and we shall henceforth drop the tildes.

The subgroup of $Aut(G_6)$ generated by $\{a, b, c, d, e, f\}$ is normal, and is a semidirect product $Z^3 \rtimes (Z/2Z)^3$ with presentation

$$\begin{aligned} \langle a, b, c, d, e, f \mid a^2 = b^2 = c^2 = 1, a, b, c \text{ commute}, d, e, f \text{ commute}, \\ ada = d^{-1}, ae = ea, af = fa, bd = db, beb = e^{-1}, bf = fb, \\ cd = dc, ce = ec, cfc = f^{-1} \rangle. \end{aligned}$$

This subgroup contains the inner automorphisms $c_x = bcd$, $c_y = acef$ and $c_z = c_x c_y$ determined by conjugation by x and y . In particular, $c_x^2 = d^2$, $c_y^2 = e^2$ and $c_z^2 = f^2$. Adjoining the generator j gives another normal subgroup, in which $j^3 = abce$, so $j^6 = 1$, and j acts on $\langle a, b, c, d, e, f \rangle$ as follows:

$$\begin{aligned} jaj^{-1} = c, jbj^{-1} = ad^{-1}, jcj^{-1} = be, jdj^{-1} = f, \\ jej^{-1} = d, jfj^{-1} = e^{-1}. \end{aligned}$$

This subgroup has index 2 in $Aut(G_6)$. The remaining generator i is an involution ($i^2 = 1$), and there are further relations

$$idi = e, iei = d, ifi = f^{-1}, iai = b, ibi = a, ici = cf, jiji = d.$$

Therefore $Out(G_6)$ has the presentation

$$\begin{aligned} \langle a, b, c, e, i, j \mid a^2 = b^2 = c^2 = e^2 = i^2 = j^6 = 1, a, b, c, e \text{ commute}, \\ iai = b, ici = ae, jaj^{-1} = c, jbj^{-1} = abc, jcj^{-1} = be, jej^{-1} = bc, \\ j^3 = abce, (ji)^2 = bc \rangle. \end{aligned}$$

(The images of d and f are represented by bc and ace , respectively.) The natural homomorphism from $Out(G_6)$ to $Aut(G_6/T) \cong GL(2, \mathbb{F}_2)$ is onto, as the images of i and j generate $GL(2, \mathbb{F}_2)$, and its kernel is the subgroup generated by $\{a, b, c, e\}$. Thus $Out(G_6)$ has order 96.

Comparison with [18]. As the group element labeled z and the automorphisms labeled a, b, c by Zimmermann differ from ours, we shall add the subscript “ Z ” for clarity. The presentation for G_6 used in [18] reduces to

$$\langle x, y, z_Z \mid xy^2x^{-1}y^2 = yx^2y^{-1}x^2 = 1, z_Zyx = x^2z_Z^2 = z_Z^2x^2 \rangle.$$

Thus $z_Z = yx^{-1}$, so $z_Z = y^2z^{-1}$, and $z_Z^2 = z^{-2}$. His choice of representatives for a generating set for $Out(G_6)$ is $\{a_Z, b_Z, c_Z, I, S, T\}$, where $a_Z = d^{-1}$, $b_Z = e^{-1}$, $c_Z = f^{-1}$, $I = xade$, $S = ideab$ and $T = j^{-1}i^{-1}xadf$. He observes also that $Out(G_6)$ is an extension of $S_3 \times Z/2Z$ by the normal subgroup $(Z/2Z)^3$ generated by $\{d, e, f\}$, but the extension does not split, since the centre of $Out(G_6)$ is generated by the image of the involution ab , and thus is too small.

7. 2-KNOTS WITH $\pi' \cong G_6$ ARE NOT REFLEXIVE

Since G_6 is solvable and $H_1(G_6) \cong (Z/4Z)^2$, an automorphism (v, A) of G_6 is meridional if and only if its image in $\text{Aut}(G_6/T) \cong GL(2, 2)$ has order 3. Thus its image in $\text{Out}(G_6)$ is conjugate to $[j]$, $[j]^{-1}$, $[ja]$ or $[jb]$. The latter pair are orientation-preserving and each is conjugate to its inverse (via $[i]$). However $(ja)^3 = 1$ while $(jb)^3 = de^{-1}f$, so $[jb]^3 = [ab] \neq 1$. Thus $[ja]$ is not conjugate to $[jb]^\pm$, and the knot groups $G(+) = G_6 \rtimes_{[ja]} Z$ and $G(-) = G_6 \rtimes_{[jb]} Z$ are distinct. The corresponding knot manifolds are the mapping tori of the isometries of HW determined by $[ja]$ and $[jb]$, and are flat 4-manifolds.

We may use the geometry of flat manifolds to adapt the argument of Lemma 18.3 of [5] to our present situation, as follows.

Theorem 3. *Let K be a 2-knot with group $G(+)$ or $G(-)$. Then K is not reflexive.*

Proof. The knot manifold $M = M(K)$ is a flat 4-manifold, $M \cong \mathbb{R}^4/\pi$ say, by Theorem 8.1 and the discussion in Chapter 16 of [5]. The weight orbit of K may be represented by a geodesic simple closed curve C through the basepoint P of \mathbb{R}^4/π . Let γ be the image of C in π .

Let h be a self-homeomorphism of \mathbb{R}^4/π which fixes C pointwise. Since M is aspherical h is based-homotopic to an affine diffeomorphism α , and then $\alpha_*(\gamma) = h_*(\gamma) = \gamma$. Let $\widehat{M} \cong \mathbb{R}^3 \times S^1$ be the covering space corresponding to the subgroup $\langle \gamma \rangle \cong Z$, and fix a lift \widehat{C} . A homotopy from h to α lifts to a proper homotopy between the lifts \widehat{h} and $\widehat{\alpha}$ to self-homeomorphisms fixing \widehat{C} . Now the behaviour at ∞ of these maps is determined by the behaviour near the fixed point sets, as in [1, 7] or Lemma 18.3 of [5]. Since the affine diffeomorphism $\widehat{\alpha}$ does not change the framing of the normal to \widehat{C} it follows that \widehat{h} and h do not change the normal framings either. \square

8. SYMMETRIES OF 2-KNOTS WITH GROUP $G(+)$

The orthogonal matrix $-JX$ is a rotation through $\frac{2\pi}{3}$ about the axis in the direction $e_1 + e_2 - e_3$. The fixed point set of the isometry $[ja]$ of HW is the image of the line $\lambda(s) = s(e_1 + e_2 - e_3) - \frac{1}{4}e_2$. The knots corresponding to the canonical section are the 3-twist spin of the figure eight knot $\tau_3 4_1$ and its Gluck reconstruction $\tau_3 4_1^*$. The knot $\tau_3 4_1$ is \pm amphicheiral and invertible [12]. We shall show that $\tau_3 4_1$ is *strongly* \pm amphicheiral, but not strongly invertible. We shall also show that none of the other 2-knots with group $G(+)$ are amphicheiral or invertible.

Theorem 4. *Let $\pi = G(+)$. Then every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $x^{2n}t$.*

Proof. If $t \in \pi$ represents a generator of $\pi/\pi' \cong Z$ it is a weight element, since π is solvable. Suppose that $c_t = ja$. If u is another weight element with $[c_u] = [ja]$ then c_u is conjugate in $\text{Aut}(G_6)$ to c_{gt} , for some $g \in \pi'' = G'_6$, by Theorem 14.1 of [5]. Suppose that $g = x^{2m}y^{2n}z^{2p}$. Let $\lambda(g) = m + n - p$ and $w = x^{2n}y^{2p}$. Then $w^{-1}gtw = x^{2\lambda(g)}t$. On the other hand, if $\psi \in \text{Aut}(G_6)$ then $\psi c_{gt}\psi^{-1} = c_{ht}$ for some $h \in G'_6$ if and only if the images of ψ and ja in $\text{Aut}(G_6)/G'_6$ commute. If so, ψ is in the subgroup generated by $\{def^{-1}, jb, ce\}$. It is easily verified that $\lambda(h) = \lambda(g)$ for any such ψ . (It suffices to check this for the generators.)

Thus $x^{2n}t$ is a weight element representing $[ja]$, for all $n \in \mathbb{Z}$, and $x^{2m}t$ and $x^{2n}t$ are in the same strict weight orbit if and only if $m = n$. \square

Note that λ is given by dot product with the axis of $-JX$.

Lemma 5. *If $n = 0$ then $C_{\text{Aut}(G_6)}(ja)$ and $N_{\text{Aut}(G_6)}(\langle ja \rangle)$ are generated by $\{ja, def^{-1}, abce\}$ and $\{ja, ice, abce\}$, respectively. The subgroup which preserves the orientation of \mathbb{R}^3 is generated by $\{ja, ice\}$.*

If $n \neq 0$ then $N_{\text{Aut}(G_6)}(\langle d^{2n}ja \rangle) = C_{\text{Aut}(G_6)}(d^{2n}ja)$ and is generated by $\{d^{2n}ja, def^{-1}\}$. This subgroup acts orientably on \mathbb{R}^3 .

Proof. This is straightforward. (Note that $abce = j^3$ and $def^{-1} = (ice)^2$.) \square

Lemma 6. *The mapping torus $M([ja])$ has an orientation reversing involution which fixes a canonical section pointwise, and an orientation reversing involution which fixes a canonical section setwise but reverses its orientation. There is no orientation preserving involution of M which reverses the orientation of any section.*

Proof. Let $\omega = abcd^{-1}f = abce(ice)^{-2}$, and let $p = \lambda(\frac{1}{4}) = \frac{1}{4}(e_1 - e_3)$. Then $\omega = (2p, -I_3)$, $\omega^2 = 1$, $\omega ja = ja\omega$ and $\omega(p) = ja(p) = p$. Hence $\Omega = m([\omega])$ is an orientation reversing involution of $M([ja])$ which fixes the canonical section determined by the image of p in HW .

Let $\Psi([f, s]) = [[iab](f), 1 - s]$ for all $[f, s] \in M([ja])$. This is well-defined, since $(iab)ja(iab)^{-1} = (ja)^{-1}$, and is an involution, since $(iab)^2 = 1$. It is clearly orientation reversing, and since $iab(\lambda(\frac{1}{8})) = \lambda(\frac{1}{8})$ it reverses the section determined by the image of $\lambda(\frac{1}{8})$ in HW .

On the other hand, $\langle ja, ice \rangle \cong Z/3Z \rtimes_{-1} Z$, and the elements of finite order in this group do not invert ja . \square

Theorem 7. *Let K be a 2-knot with group $G(+)$ and weight element $u = x^{2n}t$, where t is the canonical section. If $n = 0$ then K is strongly \pm amphicheiral, but is not strongly invertible. If $n \neq 0$ then K is neither amphicheiral nor invertible.*

Proof. Suppose first that $n = 0$. Since $-JX$ has order 3 it is conjugate in $GL(3, \mathbb{R})$ to a block diagonal matrix $\Lambda(-JX)\Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R(\frac{2\pi}{3}) \end{pmatrix}$, where $R(\theta) \in GL(2, \mathbb{R})$ is rotation through θ . Let $R_s = R(\frac{2\pi}{3}s)$ and $\xi(s) = ((I_3 - A_s)p, A_s)$, where $A_s = \Lambda^{-1} \begin{pmatrix} 1 & 0 \\ 0 & R_s \end{pmatrix} \Lambda$, for $s \in \mathbb{R}$. Then ξ is a 1-parameter subgroup of $Aff(3)$, such that $\xi(s)(p) = p$ and $\xi(s)\omega = \omega\xi(s)$ for all s . In particular, $\xi|_{[0,1]}$ is a path from $\xi(0) = 1$ to $\xi(1) = ja$ in $Aff(3)$. Let $\Xi : \mathbb{R}^3 \times S^1 \rightarrow M(ja)$ be the homeomorphism given by $\Xi(v, e^{2\pi is}) = [\xi(v), s]$ for all $(v, s) \in \mathbb{R}^3 \times [0, 1]$. Then $\Xi^{-1}\Omega\Xi = \omega \times id_{S^1}$ and so Ω does not change the framing. Therefore K is strongly $-$ amphicheiral.

Similarly, if we let $\zeta(s) = ((I_3 - A_s)\lambda(\frac{1}{8}), A_s)$ then $\zeta(s)(\lambda(\frac{1}{8})) = \lambda(\frac{1}{8})$ and $iab\zeta(s)iab = \zeta(s)^{-1}$, for all $s \in \mathbb{R}$, and $\zeta|_{[0,1]}$ is a path from 1 to ja in $Aff(3)$. Let $Z : \mathbb{R}^3 \times S^1 \rightarrow M(ja)$ be the homeomorphism given by $Z(v, e^{2\pi is}) = [\zeta(s)(v), s]$ for all $(v, s) \in \mathbb{R}^3 \times S^1$. Then

$$\begin{aligned} Z^{-1}\Psi Z(v, z) &= (\zeta(1-s)^{-1}iab\zeta(s)(v), z^{-1}) \\ &= (\zeta(1-s)^{-1}\zeta(s)^{-1}iab(v), z^{-1}) = ((ja)^{-1}iab(v), z^{-1}) \end{aligned}$$

for all $(v, z) \in \mathbb{R}^3 \times S^1$. Hence Ψ does not change the framing, and so K is strongly $+$ amphicheiral. However it is not strongly invertible, by Lemma 6.

If $n \neq 0$ every such self-homeomorphism h preserves the orientation and fixes the meridian, by Lemma 5, and so K is neither amphicheiral nor invertible. \square

9. SYMMETRIES OF 2-KNOTS WITH GROUP $G(-)$

A similar analysis applies when the knot group is $G(-)$, i.e., when the meridional automorphism is $jb = (\frac{1}{4}(e_1 - e_2), -JY)$. (The orthogonal matrix $-JY$ is now a rotation through $\frac{2\pi}{3}$ about the axis in the direction $e_1 - e_2 + e_3$.) All 2-knots with group $G(-)$ are fibred, and the characteristic map $[jb]$ has finite order, but none of these knots are twist-spins, as we shall show below.

Theorem 8. *Let $\pi = G(-)$. Then every strict weight orbit representing a given generator t for π/π' contains an unique element of the form $x^{2n}t$.*

Proof. The proof is very similar to that of Theorem 4. The main change is that we should define the homomorphism λ by $\lambda(x^{2m}y^{2n}z^{2p}) = m - n + p$. \square

Corollary. *No 2-knot with group $G(-)$ is a twist-spin.*

Proof. Suppose that $G(-)$ is the group of the r -twist-spin of a classical knot. Then the r th power of a meridian is central. The power $(x^{2n}t)^r$ is central in $G(-)$ if and only if $(d^{2n}jb)^r = 1$ in $\text{Aut}(G_6)$. But $(d^{2n}jb)^3 = d^{2n}f^{2n}e^{-2n}(jb)^3 = (de^{-1}f)^{2n+1}$. Therefore $d^{2n}jb$ has infinite order, and so $G(-)$ is not the group of a twist-spin. \square

Lemma 9. *If $n = 0$ then $C_{\text{Aut}(G_6)}(jb)$ and $N_{\text{Aut}(G_6)}(\langle jb \rangle)$ are generated by $\{jb\}$ and $\{jb, i\}$, respectively. If $n \neq 0$ then $N_{\text{Aut}(G_6)}(\langle d^{2n}jb \rangle) = C_{\text{Aut}(G_6)}(d^{2n}jb)$ and is generated by $\{d^{2n}jb, de^{-1}f\}$. These subgroups act orientably on \mathbb{R}^3 .* \square

The isometry $[jb]$ has no fixed points in $G_6 \setminus \mathbb{R}^3$. We shall define a preferred section as follows. Let $\gamma(s) = \frac{2s-1}{8}(e_1 - e_2) - \frac{1}{8}e_3$, for $s \in \mathbb{R}$. Then $\gamma(1) = jb(\gamma(0))$, and so $\gamma|_{[0,1]}$ defines a section of $p_{[jb]}$. We shall let the image of $(\gamma(0), 0)$ be the basepoint for $M([jb])$.

Theorem 10. *Let K be a 2-knot with group $G(-)$ and weight element $u = x^{2nt}$, where t is the canonical section. If $n = 0$ then K is strongly +amphicheiral but not invertible. If $n \neq 0$ then K is neither amphicheiral nor invertible.*

Proof. Suppose first that $n = 0$. Since $i(\gamma(s)) = \gamma(1-s)$ for all $s \in \mathbb{R}$ the section defined by $\gamma|_{[0,1]}$ is fixed setwise and reversed by the orientation reversing involution $[f, s] \mapsto [[i](f), 1-s]$. Let B_s be a 1-parameter subgroup of $O(3)$ such that $B_1 = jb$. Then we may define a path from 1 to jb in $\text{Aff}(3)$ by setting $\zeta(s) = ((I_3 - B_s)\gamma(s), B_s)$ for $s \in \mathbb{R}$. We see that $\zeta(0) = 1$, $\zeta(1) = jb$, $\zeta(s)(\gamma(s)) = \gamma(s)$ and $i\zeta(s)i = \zeta(s)^{-1}$ for all $0 \leq s \leq 1$. As in Theorem 5 it follows that the involution does not change the framing and so K is strongly +amphicheiral.

The other assertions follow from Lemma 9, as in Theorem 7. \square

In particular, only $\tau_3 4_1, \tau_3 4_1^*$ and the knots obtained by surgery on the section of $M([jb])$ defined by $\gamma|_{[0,1]}$ admit orientation-changing symmetries.

10. NORMAL FORMS FOR MERIDIANAL AUTOMORPHISMS OF $\Gamma(e, \eta)$

Let $M(e, \eta)$ be the 2-fold branched covering of S^3 , branched over a Montesinos knot $k(e, \eta) = K(0|e; (3, \eta), (3, 1), (3, 1))$, with e even

and $\eta = \pm 1$. This 3-manifold is Seifert fibred over the flat 2-orbifold $S(3, 3, 3)$, and $\Gamma(e, \eta) = \pi_1(M(e, \eta))$ has a presentation

$$\langle h, x, y, z \mid x^3 = y^3 = z^{3\eta} = h, xyz = h^e \rangle,$$

for some $\eta = \pm 1$. Let $u = z^{-1}x$, $v = xz^{-1}$ and $q = 3e - \eta - 2$. Then $\Gamma(e, \eta)$ also has the presentation

$$\langle u, v, z \mid zuz^{-1} = v, zvz^{-1} = v^{-1}u^{-1}z^{3\eta-3}, uvv^{-1}u^{-1} = z^{3\eta q} \rangle.$$

The image of $z^{3\eta}$ in $\Gamma(e, \eta)$ generates the centre $\zeta\Gamma(e, \eta)$, and $P = \Gamma(e, \eta)/\zeta\Gamma(e, \eta)$ is the orbifold fundamental group of $S(3, 3, 3)$.

An automorphism ϕ of $\Gamma(e, \eta)$ must preserve characteristic subgroups such as the centre $\zeta\Gamma(e, \eta)$ (generated by z^3) and the maximal nilpotent normal subgroup $\sqrt{\Gamma(e, \eta)}$ (generated by u, v and z^3). Let F be the subgroup of $\text{Aut}(\Gamma(e, \eta))$ consisting of automorphisms which induce the identity on $\Gamma(e, \eta)/\sqrt{\Gamma(e, \eta)} \cong Z/3Z$ and $\sqrt{\Gamma(e, \eta)}/\zeta\Gamma(e, \eta) \cong Z^2$. Automorphisms in F also fix the centre, and are of the form $k_{m,n}$, where

$$k_{m,n}(u) = uz^{3\eta s}, \quad k_{m,n}(v) = vz^{3\eta t} \quad \text{and} \quad k_{m,n}(z) = z^{3\eta p+1}u^m v^n,$$

for $(m, n) \in \mathbb{Z}^2$. These formulae define an automorphism if and only if $s-t = -nq$, $s+2t = mq$ and $6p = (m+n)((m+n-1)q+2(\eta-1))$.

In particular, conjugation by u and v give $c_u = k_{-2,-1}$ and $c_v = k_{1,-1}$, respectively. If $\eta = 1$ then $q = 3e$, so $s = (m-2n)e$, $t = -(m+n)e$ and $p = \binom{m+n}{2}e$ are integers for all $m, n \in \mathbb{Z}$. In this case $F \cong \mathbb{Z}^2$ is generated by $k = k_{1,0}$ and c_u . If $\eta = -1$ then $m+n \equiv 0 \pmod{3}$, and F is generated by c_u and c_v . In this case F has index 3 in \mathbb{Z}^2 .

We may define automorphisms b and r by the formulae:

$$\begin{aligned} b(u) &= v^{-1}z^{3\eta e-3}, \quad b(v) = uvz^{3\eta(e-1)} \quad \text{and} \quad b(z) = z; \quad \text{and} \\ r(u) &= v^{-1}, \quad r(v) = u^{-1} \quad \text{and} \quad r(z) = z^{-1}. \end{aligned}$$

It is easily checked that $b^6 = r^2 = (br)^2 = 1$ and that conjugation by z gives $c_z = b^4$. Since $\Gamma(e, \eta)/\Gamma(e, \eta)'$ is finite, $\text{Hom}(\Gamma(e, \eta), \zeta\Gamma(e, \eta)) = 0$, and so the natural homomorphism from $\text{Aut}(\Gamma(e, \eta))$ to $\text{Aut}(P)$ is injective. If $\eta = +1$ this homomorphism is an isomorphism, and $\text{Aut}(\Gamma(e, 1))$ has a presentation

$$\begin{aligned} \langle b, c_u, k, r \mid b^6 = r^2 = (br)^2 = 1, \quad c_u k &= k c_u, \quad b c_u b^{-1} = c_u^{-1} k^{-3}, \\ b k b^{-1} &= c_u k^2, \quad r c_u r = c_u k^3, \quad r k r = k^{-1} \rangle. \end{aligned}$$

(Here $c_v = c_u k^3$.) On the other hand, $\text{Aut}(\Gamma(e, -1))$ has a presentation

$$\begin{aligned} \langle b, c_u, c_v, r \mid b^6 = r^2 = (br)^2 = 1, \quad c_u c_v &= c_v c_u, \quad b c_u b^{-1} = c_v^{-1}, \\ b c_v b^{-1} &= c_u c_v, \quad r c_u r = c_v, \quad r c_v r = c_u \rangle. \end{aligned}$$

Hence $\text{Out}(\Gamma(e, 1)) \cong S_3 \times Z/2Z$, while $\text{Out}(\Gamma(e, -1)) \cong (Z/2Z)^2$.

In each case an automorphism ϕ is meridional if and only if $[\phi]$ is conjugate to $[r]$, and so there is an unique corresponding knot group $\pi(e, \eta) = \Gamma(e, \eta) \rtimes_r Z$.

11. EMBEDDINGS IN THE AFFINE GROUP

The group P embeds as a discrete subgroup of $Isom(\mathbb{E}^2)$, via $u \mapsto (e_1, I_2)$, $v \mapsto (e_2, I_2)$ and $z \mapsto (0, -\beta)$, where $\beta = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. The images of u and v in P form a basis for the translation subgroup $T(P) \cong Z^2$, and $P = T(P) \rtimes_{-\beta} (Z/3Z)$. It is easily seen that $C_{Aff(2)}(P) = 1$, and so $Aut(P) \cong N_{Aff(2)}(P)$. If $(v, A) \in N_{Aff(2)}(P)$ then $(I_2 + \beta)v \in Z^2$ and A is in the subgroup D of $GL(2, \mathbb{R})$ generated by the matrices β and $\rho = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, which has order 12. Thus $Aut(P) = (I_2 + \beta^{-1})T(P) \rtimes D$. Hence $Out(P) \cong D \cong S_3 \times Z/2Z$, where the first factor is generated by the images of u and $(0, -I_2)$ and the second factor is generated by the image of $(0, -\rho)$.

Let $Nil < GL(3, \mathbb{R})$ be the group of 3×3 upper triangular matrices

$$[x, y, w] = \begin{pmatrix} 1 & x & w \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and let $Aut(Nil)$ be the group of Lie automorphisms. As a set, $Aut(Nil)$ is the cartesian product $GL(2, \mathbb{R}) \times \mathbb{R}^2$, with $(A, \mu) = \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, (\mu_1, \mu_2)\right)$ acting via $(A, \mu)([x, y, w]) =$

$$[ax + cy, bx + dy, \mu_1 x + \mu_2 y + (ad - bc)w + bcxy + \frac{ab}{2}x(x-1) + \frac{cd}{2}y(y-1)].$$

All such automorphisms are orientation preserving. The product of (A, μ) with $(B, \nu) = \left(\begin{pmatrix} g & j \\ h & k \end{pmatrix}, (n_1, n_2)\right)$ is

$$(A, \mu) \circ (B, \nu) = (AB, \mu B + \det(A)\nu + \frac{1}{2}\eta(A, B)),$$

where

$$\eta(A, B) = (abg(1-g) + cdh(1-h) - 2bcgh, abj(1-j) + cdk(1-k) - 2bcjk).$$

Let $Aff(Nil) = Nil \rtimes Aut(Nil)$. Then $Aff(Nil)$ acts on the open 3-manifold $Nil \cong \mathbb{R}^3$ by $(n, \sigma)(n') = n\sigma(n')$. The abelianization $Nil \rightarrow \mathbb{R}^2 = Nil/\zeta Nil$ extends to an epimorphism $p : Aff(Nil) \rightarrow Aff(2)$, given by $p(n, A, \mu) = \left(\begin{pmatrix} x \\ y \end{pmatrix}, A\right)$ for $n = [x, y, w] \in Nil$, $A \in GL(2, \mathbb{R})$ and $\mu \in \mathbb{R}^2$. We may embed $\Gamma(e, \eta)$ in $Aff(Nil)$ by

$$u \mapsto ([1, 0, 0], \iota), \quad v \mapsto ([0, 1, 0], \iota) \quad \text{and} \quad z \mapsto ([0, 0, \frac{-1}{3q}], \alpha),$$

where $\iota = id_{Nil}$ and $\alpha = (-\beta, (0, \frac{\eta-1}{q}))$. (Note that $vuv^{-1}u^{-1} \mapsto ([0, 0, -1], \iota)$.) Let $N = N_{Aff(Nil)}(\Gamma(e, \eta))$ and $C = C_{Aff(Nil)}(\Gamma(e, \eta))$. As in the flat case, $Aut(\Gamma(e, \eta)) \cong N/C$ and $Out(\Gamma(e, \eta)) \cong N/CT(e, \eta)$.

It is easily seen that $C = \zeta Nil = \{([0, 0, z], \iota) \mid z \in \mathbb{R}\}$. If $n = [x, y, w]$ and $(n, A, \mu) \in N$ then $((\frac{x}{y}), A) \in N_{Aff(2)}(P)$, so $A \in D$ and $(\frac{x}{y}) \in (I_2 + \beta)^{-1}\mathbb{Z}^2$. If $A = I_2$ then (n, I_2, μ) is in N if and only if it normalizes $\sqrt{\Gamma(e, \eta)}$ and $(n, I_2, \mu)z = (n', \iota)z(n, I_2, \mu)$ for some $n' \in \sqrt{\Gamma(e, \eta)}$. The latter condition implies that $(I_2, \mu)\alpha = \alpha(I_2, \mu)$, and so $\mu(\beta + I_2) = 0$. Thus we must have $\mu = 0$ and $(I_2, \mu) = \iota$. The remaining conditions then imply that $x, y \in \frac{1}{q}\mathbb{Z}$. If $\eta = 1$ (so $q = 3e$) this is satisfied by all $(\frac{x}{y}) \in (I_2 + \beta)^{-1}\mathbb{Z}^2 < \frac{1}{3}\mathbb{Z}^2$. If $\eta = -1$ then $x, y \in \mathbb{Z}$. Thus the natural map from $Aut(\Gamma(e, \eta))$ to $Aut(P)$ is an isomorphism if $\eta = 1$, and has image of index 3 if $\eta = -1$.

12. 2-KNOTS WITH GROUP $\pi(e, \eta)$

Let $R = ([0, 0, 0], \rho, (0, 0))$ in $Aff(Nil)$. Then $R^2 = 1$ and $R([x, y, z]) = [-y, -x, -z]$ for all $[x, y, z] \in Nil$. The fixed point set of the action of R on Nil is the connected curve $\{[s, -s, 0] \mid s \in \mathbb{R}\}$. Thus the fixed point set of the involution $[R]$ of $M(e, \eta)$ induced by R is connected and nonempty. The corresponding 2-knot is $\tau_2 k(e, \eta)$. This is reflexive and +amphicheiral [12].

Theorem 11. *The knot $K = \tau_2 k(e, \eta)$ is reflexive and strongly +amphicheiral, but is not invertible.*

Proof. Let $S([m, s]) = [b^3(m), s]$ and $h([m, s]) = [m, 1 - s]$ for $m \in M(e, \eta)$ and $0 \leq s \leq 1$. Then S and h define commuting involutions of $M([R])$, which each fix the canonical section setwise.

As remarked in §3, in order to determine how these involutions affect the framing we may pass to the irregular covering space $M(R) = Nil \times_R S^1$. We shall identify the space Nil with \mathbb{R}^3 , in the obvious way.

Let $R(\theta) \in GL(2, \mathbb{R})$ be rotation through θ , and let $P = \begin{pmatrix} R(\frac{\pi}{4}) & 0 \\ 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R})$. Then $PRP^{-1} = diag[1, -1, -1]$. We may isotope PRP^{-1} back to the identity, via $Q_s = \begin{pmatrix} 1 & 0 \\ 0 & R(s\pi) \end{pmatrix}$, for $0 \leq s \leq 1$. Let $Q : \mathbb{R}^3 \times S^1 \rightarrow M(PR P^{-1})$ be the homeomorphism given by $Q(v, e^{2\pi i s}) = [Q_s(v), s]$ for all $(v, s) \in \mathbb{R}^3 \times [0, 1]$. Then $Q^{-1}hQ((v, z) = (Q_{2s-1}(v), z^{-1})$ for all $(v, z) \in \mathbb{R}^3 \times S^1$. After reversing the S^1 factor this is just the twist, and so h changes the framing. Thus K is reflexive.

The automorphism b^3 acts linearly, via $b^3([x, y, z]) = [-x, -y, z + (e\eta - 1)(x + y)]$, and so $Pb^3P^{-1} = \begin{pmatrix} -I_2 & 0 \\ \mu & 1 \end{pmatrix}$, where $\mu = (e\eta - 1, e\eta -$

1) $R(-\frac{\pi}{4})$. We may isotope Pb^3P^{-1} to $d = \text{diag}[-1, -1, 1]$ through invertible matrices which commute with PRP^{-1} . Let $D([v, s]) = [d(v), s]$. Then S and D twist the framing in the same way. Since $Q^{-1}DQ(v, e^{2\pi is}) = (Q_{-s}dQ_s(v), e^{2\pi is}) = (dQ_{2s}(v), e^{2\pi is})$, for all $(v, s) \in \mathbb{R}^3 \times [0, 1]$, it follows that S changes the framing.

The composite Sh is an involution which reverses the orientation and the meridian, but does not twist the framing. Hence K is strongly +amphicheiral.

Since automorphisms of $\Gamma(e, \eta)$ are orientation preserving K is not -amphicheiral or invertible. \square

The other knots with such groups are less symmetrical.

Theorem 12. *Let K be a 2-knot with $\pi = \pi K \cong \pi(e, \eta)$. Then K is reflexive, and every strict weight orbit representing the canonical section t for π/π' contains a unique element of the form u^nt . If $n \neq 0$ then $N_{\text{Aut}(\Gamma(e, \eta))}(\langle u^nr \rangle) = C_{\text{Aut}(\Gamma(e, \eta))}(u^nr) = \langle u^nr, uv^{-1} \rangle$, and u^nr is not conjugate to its inverse. Hence K is neither amphicheiral nor invertible.*

Proof. The first assertion follows from Lemma 1.

Recall that F is the group of automorphisms of $\Gamma(e, \eta)$ which induce the identity on $\Gamma(e, \eta)/\sqrt{\Gamma(e, \eta)}$ and $\sqrt{\Gamma(e, \eta)}/\zeta\Gamma(e, \eta)$. If $\psi r \psi^{-1} = rk$ for some $k \in F$ then we may assume that $\psi \in F$, and then $k \in (I - \rho)F$. The parametrization of the weight orbits follows by the argument of Theorem 4, with minor changes. (Note that $t \mapsto th$ defines an automorphism of π .)

If $n \neq 0$ then u^nt is not conjugate to its inverse. Hence K is not +amphicheiral, and since automorphisms of $\Gamma(e, \eta)$ are orientation preserving K is neither -amphicheiral nor invertible. \square

13. DOUBLE NULL CONCORDANCE

A 2-knot K is doubly null-concordant (or doubly slice) if $K = S^4 \cap U$, where S^4 is embedded as the equator of S^5 and U is a trivial 3-knot. An equivalent condition is that $M(K)$ embeds in $S^1 \times S^{n+2}$, via a map which induces the abelianization on πK . Thus whether K is doubly null concordant depends only on (the h -cobordism class of) $M(K)$.

The knot $k(0, -1) = 9_{46}$ is doubly null-concordant [8, 17], and hence so are all of its twist spins. Since $M(\tau_2 k(0, -1))$ is determined up to homeomorphism by its group, every knot with group $\pi(0, -1)$ must be doubly null-concordant.

The Farber-Levine pairing of a doubly null-concordant knot is hyperbolic, and so the torsion submodule of π'/π'' is a direct double.

Moreover, if F is a field then $H^*(M(K)'; F)$ splits as a sum of graded subalgebras $A^* \oplus B^*$, with $A^i \cup A^{3-i} = B^i \cup B^{3-i} = 0$ for $i = 1$ or 2 , while duality gives a perfect pairing of A^i with B^{3-i} [11].

The Alexander polynomials of Fox's knot, the Cappell-Shaneson 2-knots and the knots with $\pi' \cong \Gamma_q$ for some odd $q \geq 1$ are all irreducible and nonconstant. For such knots $H^*(M(K)'; \mathbb{Q})$ cannot split as above. If $\pi \cong G(\pm)$ then $\pi'/\pi'' \cong (Z/4Z)^2$ is finite. However it is cyclic as a Λ -module, and so the cohomology ring with coefficients \mathbb{F}_2 does not split. If $\pi = \pi(e, \eta)$ and $|q| = |3e - \eta - 2| \neq 1$ then $\pi'/\pi'' \cong Z/3qZ \oplus Z/3Z$ is not a direct double, and so the Farber-Levine pairing is not hyperbolic. Thus none of these knots are doubly null-concordant.

Similarly, knots whose groups have finite commutator subgroup other than I^* are not doubly slice, since their Farber-Levine pairings are not hyperbolic. All 2-knots with perfect commutator subgroup are topologically doubly slice. However $\tau_5 3_1$ is not *smoothly* doubly slice [16]. (Crude estimates based on the surgery exact sequence show that there are infinitely many such knots.)

The classification of these knots up to *stable* DNC equivalence is much more subtle, and we shall not attempt this here.

All fibred 2-knots may be realized as smooth knots in a smooth homotopy 4-sphere. Twist spins (such as $\tau_3 4_1$ and $\tau_2 k(e, \eta)$) are smooth knots in the standard smooth structure on S^4 . In [4] it is shown that this holds also for many Cappell-Shaneson 2-knots. Is this so for the other knots with torsion-free solvable groups?

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